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ON ALMOST PERIODIC MOVEMENTS

BY

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Introduction.

Let $\mathfrak{M}$ denote a metric space. The points of $\mathfrak{M}$ are denoted by $x, y, \cdots$ and the distance by $[x, y]$. A continuous function $x = f(t)$, $-\infty < t < \infty$, $x \in \mathfrak{M}$, will be called a movement in $\mathfrak{M}$. An almost periodic movement in $\mathfrak{M}$ is defined in close analogy to the classical complex-valued almost periodic functions introduced by H. Bohr [3], [5]. If $x = g(u_1, \cdots, u_m)$, $-\infty < u_\nu < \infty$, $\nu = 1, \cdots, m$; $x \in \mathfrak{M}$, is a continuous function with the period 2π in each variable, and $\beta_1, \cdots, \beta_m$ are rationally independent real numbers, the function $x = g(\beta_1 t, \cdots, \beta_m t)$ will be called a diagonal movement.

In Chapter 1 we shall investigate whether or not every continuous movement in $\mathfrak{M}$ can be approximated uniformly to any given accuracy by a diagonal movement. It will be proved that the answer is affirmative if $\mathfrak{M}$ is a complete space with the following property: To every compact subset $\mathfrak{C}$ of $\mathfrak{M}$ corresponds a number Δ, such that any two points $x, y \in \mathfrak{C}$ with $[x, y] \leqq \Delta$ can be connected in $\mathfrak{M}$ by a continuous curve which depends continuously on x, y and which reduces to the point x if $y = x$. A space with this property will be called continuously locally arcwise connected. We shall also prove that a uniformly continuous family $f(t; v)$ of almost periodic movements can be approximated uniformly to any given accuracy by a uniformly continuous family of diagonal movements. These results will be proved by methods very similar to those applied for ordinary almost periodic functions. On the other hand, it will be proved that certain almost periodic movements in the so-called solenoidal spaces introduced by D. van Dantzig [6] cannot be approximated uniformly by diagonal movements. The solenoidal spaces are complete, compact and connected, but neither locally nor arcwise connected.

Two almost periodic movements $f_1(t)$ and $f_2(t)$ are called homotopic if there exists a uniformly continuous family $f(t; v)$, $0 \leq v \leq 1$, of almost periodic movements which contains them. If the space $\mathfrak{M}$ is complete and continuously locally arcwise connected, there exists an approximating family $g(\beta_1 t, \cdots, \beta_m t; v)$ such that $f_1(t)$ and $f_2(t)$ are homotopic to the approximating diagonal functions. Thus, the homotopy between $f_1(t)$ and $f_2(t)$ gives rise to a homotopy between two continuous, periodic functions $g_1(u_1, \cdots, u_m)$ and $g_2(u_1, \cdots, u_m)$, i. e. to a homotopy between two mappings of an m-dimensional torus into $\mathfrak{M}$.

In particular, it will be proved in Chapter 2 that every almost periodic movement in a complete, continuously locally arcwise connected space is homotopic to a periodic movement if and only if every continuous mapping of a torus (of any dimension) into $\mathfrak{M}$ is homotopic to a mapping of a torus into a closed curve in $\mathfrak{M}$.

W. Fenchel and B. Jessen [7] proved that every almost periodic movement on a complete surface admitting a hyperbolic non-Euclidean metric is homotopic to a periodic movement. (In their paper it was further assumed that there are no arbitrarily short closed paths on the surface. This restriction can, however, be removed on account of the fact that an almost periodic movement is confined to a compact part of the surface.) The only topological types of surfaces which cannot be metrized in this way are the orientable and the non-orientable tori, the sphere, and the projective plane. W. Fenchel and B. Jessen proved that the result does not hold for a torus, even if deformations are allowed which do not preserve almost periodicity. For the sphere the statement is obviously true in this weaker sense, but W. Fenchel and B. Jessen considered it unlikely that every almost periodic movement on a sphere is homotopic (in the strict sense defined above) to a periodic movement. We shall prove that this conjecture is true.

The general theory of homotopy between almost periodic movements in a metric space $\mathfrak{M}$ is reduced to the theory of homotopy between mappings of m-dimensional tori into $\mathfrak{M}$.

R. H. Fox ([8], p. 509) has briefly indicated a method to put this theory into a group-theoretical form, but so far the relations between the general torus homotopy groups introduced in his paper and the ordinary homotopy groups seem not to have been investigated. It is possible, by means of the results obtained by R. H. Fox to prove that every almost periodic movement in $\mathfrak{M}$ is homotopic to a periodic movement if the ordinary homotopy groups of $\mathfrak{M}$ are trivial and all Abelian subgroups of the fundamental group of $\mathfrak{M}$ are cyclic.

CHAPTER 1.

Almost Periodic Movements in Metric Spaces.

1. Uniformly Continuous Families of Almost Periodic Movements.

In the following we shall consider a metric space $\mathfrak{M}$ consisting of points $x, y, z, \cdots$ and with a distance $[x, y]$ satisfying the conditions

1. $[x, x] = 0$; $[x, y] > 0$ when $y \neq x$.
2. $[x, y] = [y, x]$.
3. $[x, y] \leqq [x, z] + [z, x]$.

Definition 1. *A continuous movement in* $\mathfrak{M}$ *is a continuous function* $x = f(t)$, $-\infty < t < \infty$, $x \in \mathfrak{M}$. *A number* $\tau = \tau_f(\varepsilon)$ *is called a translation number of* $f(t)$ *corresponding to* $\varepsilon > 0$ (*or an* ε*-translation number*) *if the condition* $[f(t), f(t+\tau)] \leqq \varepsilon$ *is satisfied for all real values of* t. *The movement* $x = f(t)$ *is called almost periodic if the set* $\{\tau_f(\varepsilon)\}$ *of translation numbers of* $f(t)$ *corresponding to* ε *is relatively dense for every* $\varepsilon > 0$.

Definition 2. *Let* $\mathfrak{K}$ *denote a compact S-space* ([12], p. 40—45) *consisting of points denoted* $v, v_1, \cdots$ *and with neighbourhoods* $U(v)$. *A function* $x = f(t; v)$, $-\infty < t < \infty$, $v \in \mathfrak{K}$; $x \in \mathfrak{M}$ *is called a uniformly continuous family of almost periodic movements when*

1. *The function $f(t;\, v)$ is almost periodic for every fixed $v \in \mathfrak{K}$.*
2. *To $\varepsilon > 0$ and $v_0 \in \mathfrak{K}$ corresponds a neighbourhood $U_\varepsilon(v_0)$ such that $[f(t;\, v_0),\, f(t;\, v)] \leqq \varepsilon$ when $-\infty < t < \infty$, $v \in U_\varepsilon(v_0)$.*

Definition 3. *A set $\{f(t)\}$ of almost periodic movements in $\mathfrak{M}$ is called a uniformity set if there exists a real-valued almost periodic function $g(t)$ such that the set of common ε-translation numbers of all functions $f(t)$ of the set for $\varepsilon > 0$ contains the set of ε-translation numbers of $g(t)$.*

We shall now prove some elementary theorems on uniformly continuous families of almost periodic functions.

Lemma 1. *An almost periodic movement $x = f(t)$ is bounded and uniformly continuous.*

Proof. That $f(t)$ is bounded means that $x = f(t)$ remains in a bounded subdomain of $\mathfrak{M}$, i. e. that there exists a positive number K such that $[f(t_1),\, f(t_2)] \leqq K$ for all real values of t_1 and t_2. But the function $\varphi(t) = [f(0),\, f(t)]$ satisfies the condition

$$|\varphi(t_2) - \varphi(t_1)| = |[f(0),\, f(t_2)] - [f(0),\, f(t_1)]| \leqq [f(t_1),\, f(t_2)].$$

This implies that $\varphi(t)$ is almost periodic. Hence we may choose K such that $[f(0),\, f(t)] \leqq \frac{K}{2}$, for all values of t, and the first part of the lemma is proved. The proof of the second part of Lemma 1 can be copied from the proof of the corresponding property for ordinary almost periodic functions ([3] p. 30).

Lemma 2. *Let $x = f(t;\, v)$, $-\infty < t < \infty$, $v \in \mathfrak{K}$; $x \in \mathfrak{M}$ be a uniformly continuous family of almost periodic movements. There exists a constant K such that $[f(t_1;\, v),\, f(t_2;\, v)] \leqq K$ for all real values of t_1 and t_2 and for every $v \in \mathfrak{K}$.*

Proof. From Lemma 1 and from 2 in Definition 2 it follows immediately that there exists a constant $K(v_0)$ such that $[f(t_1;\, v),\, f(t_2;\, v)] \leqq K(v_0)$ when $v \in U_\varepsilon(v_0)$. But the compact S-space $\mathfrak{K}$ is covered by the neighbourhoods $U_\varepsilon(v)$, hence we can find a finite number of points $v_1, \cdots, v_n \in \mathfrak{K}$, such that $\mathfrak{K}$ is covered by $U_\varepsilon(v_1), \cdots, U_\varepsilon(v_n)$. We have then $[f(t_1;\, v),\, f(t_2;\, v)] \leqq \operatorname*{Max}_{\nu} K(v_\nu)$ for every $v \in \mathfrak{K}$. This completes the proof.

Definition 4. *The function*

$$e(\tau; v) = \underset{t}{\text{l. u. b.}}\, [f(t; v),\ f(t+\tau; v)]$$

is called the translation function of the uniformly continuous family of almost periodic movements $f(t; v)$ *and*

$$e(\tau) = \underset{v}{\text{l. u. b.}}\, e(\tau; v)$$

is called the translation majorant of $f(t; v)$.

The translation functions were introduced in the case of ordinary almost periodic functions by H. Bohr ([3], p. 37), and S. Bochner ([2], p. 136—146) used them for the study of the translation properties of almost periodic functions. Our generalized translation functions will enable us to apply theorems concerning translation numbers of ordinary almost periodic functions to almost periodic movements in $\mathfrak{M}$.

Lemma 3. *The translation function* $e(\tau; v)$ *and the translation majorant* $e(\tau)$ *are real, non-negative, and bounded.*

Proof. This lemma follows immediately from Lemma 2.

Lemma 4. *The translation function* $e(\tau; v)$ *and the translation majorant* $e(\tau)$ *satisfy the conditions*

$$e(0; u) = e(0) = 0;\ e(\tau; v) = e(-\tau; v);\ e(v) = e(-v);$$

$$e(\tau_1 + \tau_2; v) \leq e(\tau_1; v) + e(\tau_2; v);\ e(\tau_1 + \tau_2) \leq e(\tau_1) + e(\tau_2).$$

Proof. The properties in the first line are immediate consequences of Definition 4. We have further

$$e(\tau_1 + \tau_2; v) = \underset{t}{\text{l. u. b.}}\, [f(t; v),\ f(t + \tau_1 + \tau_2; v)] \leq$$

$$\underset{t}{\text{l.u.b.}}\, [f(t; v),\ f(t+\tau_1; v)] + \underset{t}{\text{l.u.b.}}\, [f(t+\tau_1; v),\ f(t+\tau_1+\tau_2; v)]$$

$$= e(\tau_1; v) + e(\tau_2; v)$$

and

$$e(\tau_1 + \tau_2) = \underset{v}{\text{l. u. b.}}\, e(\tau_1 + \tau_2; v) \leq$$

$$\underset{v}{\text{l. u. b.}}\, (e(\tau_1; v) + e(\tau_2; v)) \leq e(\tau_1) + e(\tau_2).$$

In the following it will be convenient to write $e(t; v)$ and $e(t)$ instead of $e(\tau; v)$ and $e(\tau)$.

Lemma 5. *Let v be a fixed point of $\mathfrak{K}$. The translation function $e(t; v)$ is almost periodic and the sets of translation numbers of $f(t; v)$ and of $e(t; v)$ corresponding to $\varepsilon > 0$ are identical to each other and to the set of numbers τ for which $e(\tau; v) \leqq \varepsilon$.*

Proof. A number τ is a translation number of $f(t; v)$ corresponding to $\varepsilon > 0$ if and only if $e(\tau; v) = \text{l. u. b.}\, [f(t; v), f(t+\tau; v] \leqq \varepsilon$. On the other hand we have by Lemma 4

$$(1)\quad \begin{cases} e(t+\tau; v) - e(t; v) \leqq e(\tau; v), \\ e(t; v) - e(t+\tau; v) \leqq e(-\tau; v) = e(\tau; v), \end{cases}$$

hence

$$(2)\quad e(\tau; v) = e(\tau; v) - e(0; v) = \underset{t}{\text{l. u. b.}} \left| e(t+\tau; v) - e(t; v) \right|.$$

It follows that $\left| e(t+\tau; v) - e(t; v) \right| \leqq \varepsilon$ for every t if and only if $e(\tau; v) \leqq \varepsilon$. We have thus proved that the set of translation numbers of $e(\tau; v)$ corresponding to ε is relatively dense. From Lemma 1 it follows that the set of translation numbers contains an interval about zero. Hence $e(\tau; v)$ is continuous. This completes the proof.

Lemma 6. *The translation majorant $e(t)$ is almost periodic.*

Proof. This theorem is far from trivial and its proof must be based on some deeper theorem on ordinary almost periodic functions. We prefer to make use of the theorem that a finite number of almost periodic functions have an almost periodic sum. ([3] p. 31—32).

The inequalities (1) and (2) are true for every $e(t; v)$, and also for $e(t)$. Hence, τ is a translation number of $e(t)$ corresponding to $\varepsilon > 0$ if and only if $e(\tau) \leqq \varepsilon$, i. e. if and only if $e(\tau; v) \leqq \varepsilon$ for every v in $\mathfrak{K}$. Let v be an arbitrary point of $\mathfrak{K}$ and $U_{\frac{\varepsilon}{3}}(v)$ the corresponding neighbourhood introduced in Definition 2. As $\mathfrak{K}$ is a compact S-space, we may choose a finite number of points $v_1, \cdots, v_n$ such that $\mathfrak{K} \subset U_{\frac{\varepsilon}{3}}(v_1) \cup \cdots \cup U_{\frac{\varepsilon}{3}}(v_n)$. The sum $e(t; v_1) + \cdots + e(t; v_n) = E(t)$ is an almost periodic function. If v is an arbitrary point of $\mathfrak{K}$, we can choose v_ν such that $v \in U_{\frac{\varepsilon}{3}}(v_\nu)$. It follows that

$$e(\tau; v) = \underset{t}{\text{l. u. b.}} [f(t; v), f(t+\tau; v)] \leqq$$

$$\underset{t}{\text{l. u. b.}} ([f(t; v), f(t; v_\nu)] + [f(t; v_\nu), f(t+\tau; v_\nu)] +$$

$$[f(t+\tau; v_\nu), f(t+\tau; v)]) \leqq \underset{t}{\text{l. u. b.}} \left(\frac{\varepsilon}{3} + [f(t; v_\nu), f(t+\tau; v_\nu)] + \frac{\varepsilon}{3}\right) =$$

$$e(\tau; v_\nu) + \frac{2\varepsilon}{3} \leqq E(\tau) + \frac{2\varepsilon}{3}.$$

We have thus proved that $e(\tau) \leqq \varepsilon$ if $E(\tau) \leqq \frac{\varepsilon}{3}$. But as $E(\tau) = E(\tau) - E(0)$, this condition is satisfied when τ is a translation number of the almost periodic function $E(t)$ corresponding to $\frac{\varepsilon}{3}$. This set of translation numbers is relatively dense, and it contains an interval about zero. This completes the proof.

Lemma 7. *The set of translation numbers of the translation majorant $e(t)$ corresponding to $\varepsilon > 0$ is identical to the set of common translation numbers of the almost periodic functions of the family $f(t; v)$ corresponding to the same ε.*

Proof. The set of common translation numbers of the functions $f(t; v)$ corresponding to ε is, according to Lemma 5, identical to the set of numbers τ satisfying $e(\tau; v) \leqq \varepsilon$ for every $v \in \mathfrak{R}$; but this condition is equivalent to the condition $e(\tau) \leqq \varepsilon$; and the set of numbers τ satisfying the last condition is exactly the set of translation numbers of $e(t)$ corresponding to ε.

Theorem 1. *The set of almost periodic functions belonging to a uniformly continuous family of almost periodic movements is a uniformity set.*

Proof. The theorem is an immediate corollary of Lemmas 6 and 7.

Definition 5. *For $\delta > 0$ and real numbers $\lambda_1, \cdots, \lambda_m$ the set of real numbers τ satisfying the conditions* $|\lambda_\nu \tau| < \delta \pmod{2\pi}$, $\nu = 1, \cdots, m$ *is called a $(\delta; \lambda)$-neighbourhood of zero.*

Our proof of the approximation theorem for almost periodic functions will be based on the following theorem.

Theorem 2. *The set of common ε-translation numbers of all almost periodic functions belonging to a uniformly continuous family of almost periodic movements contains a $(\delta; \lambda)$-neighbourhood of zero.*

Proof. According to Lemma 7 the set of common translation numbers is identical to the set $\{\tau_e(\varepsilon)\}$ and, according to a well-known property of ordinary almost periodic functions ([4], p. 110, Satz 2) this set contains a $(\delta; \lambda)$-neighbourhood of zero.

2. On Operations with Almost Periodic Movements in Complete Metric Spaces.

In this section we shall consider a finite number of complete metric spaces $\mathfrak{M}_1, \cdots, \mathfrak{M}_n$ and for $\nu = 1, \cdots, n$ an almost periodic movement $f_\nu(t)$ in $\mathfrak{M}_\nu$. The movement $f_\nu(t)$ can be considered as a uniformly continuous family of almost periodic movements (Definition 2) in the particular case where $\mathfrak{K}$ contains only one element. Hence, we can apply the results of Section 1.

We shall consider the topological product $\mathfrak{M} = \mathfrak{M}_1 \times \cdots \times \mathfrak{M}_n$. A point of $\mathfrak{M}$ is an ordered set $\boldsymbol{x} = (x_1, \cdots, x_n)$, $x_\nu \in M_\nu$, $\nu = 1, \cdots, n$. If $\boldsymbol{y} = (y_1, \cdots, y_n)$ is another point of $\mathfrak{M}$, the distance $[\boldsymbol{x}, \boldsymbol{y}]$ is defined by

$$[\boldsymbol{x}, \boldsymbol{y}] = \sqrt{[x_1, y_1]_1^2 + \cdots + [x_n, y_n]_n^2}$$

where $[x_\nu, y_\nu]_\nu$ is the distance between x_ν and y_ν in the metric of $\mathfrak{M}_\nu$. The space $\mathfrak{M}$ is complete.

Lemma 8. *The function $\boldsymbol{f}(t) = (f_1(t), \cdots, f_n(t))$ is an almost periodic movement in $\mathfrak{M}$.*

Proof. Let $e_\nu(\tau) = \underset{t}{\text{l. u. b.}} [f_\nu(t), f_\nu(t+\tau)]$ be the translation function of $f_\nu(t)$. According to Lemma 5 the set of translation numbers of $\boldsymbol{f}(t)$ corresponding to $\varepsilon > 0$ contains the set of all numbers τ satisfying $e_\nu(\tau) \leq \frac{\varepsilon}{\sqrt{n}}$, $\nu = 1, \cdots, n$. The function $E(t) = e_1(t) + \cdots + e_n(t)$, however, is almost periodic and $E(0) = 0$. It follows that the set of numbers τ for which $E(\tau) \leq \frac{\varepsilon}{\sqrt{n}}$, is relatively dense and contains an interval about zero. This proves the lemma.

Let $\mathfrak{M}$ be a complete metric space and $f(t)$ an almost periodic movement in $\mathfrak{M}$. The set of points of $\mathfrak{M}$ which are values of

$f(t)$ when t runs through all real numbers is called the range of $f(t)$. We shall prove the following lemma.

Lemma 9. *The closure of the range of an almost periodic movement $f(t)$ in a complete space $\mathfrak{M}$ is a compact set in $\mathfrak{M}$.*

Proof. Let $\varepsilon > 0$ be given. As $\mathfrak{M}$ is complete, we need only prove the existence of a finite number of points $x_1, \cdots, x_q$ of $\mathfrak{M}$ such that every point of the closure $\mathfrak{A}$ of the range $\mathfrak{B}$ of $f(t)$ is within ε-distance of at least one of the points x_ν. Let l be chosen such that every interval of length l contains a translation number of $f(t)$ corresponding to $\frac{\varepsilon}{3}$. According to Lemma 1 we can choose δ such that $[f(t_2), f(t_1)] \leqq \frac{\varepsilon}{3}$ when $|t_2 - t_1| \leqq \delta$. We put $x_\nu = f(\nu\delta)$, $\nu = 1, \cdots, \left[\frac{l}{\delta}\right]$. Let x be a point of $\mathfrak{A}$. We can find x' in $\mathfrak{B}$ such that $[x', x] < \frac{\varepsilon}{3}$. There exists a real number t' such that $x' = f(t')$ and a real number t'' such that $t' - t'' = \tau_f\left(\frac{\varepsilon}{3}\right)$ and $0 \leqq t'' \leqq l$. We have then $[f(t''), x'] \leqq \frac{\varepsilon}{3}$. Finally, we can choose ν such that $1 \leqq \nu \leqq \left[\frac{l}{\delta}\right]$ and $|t'' - \nu\delta| \leqq \delta$. It follows that $[f(\nu\delta), f(t'')] \leqq \frac{\varepsilon}{3}$. We have thus proved that $[x_\nu, x] < \varepsilon$ and this completes the proof.

Lemma 10. *Let $y = F(x)$, $x \in \mathfrak{M}$, $y \in \mathfrak{M}^*$, where $\mathfrak{M}$ and $\mathfrak{M}^*$ are metric spaces and $\mathfrak{M}$ is complete, be a function continuous in the closure of the range of an almost periodic movement $x = f(t)$. Then $F(f(t))$ is an almost periodic movement.*

Proof. Let $\varepsilon > 0$ be given. In consequence of Lemma 9 the function $F(x)$ is uniformly continuous in the closure $\mathfrak{A}$ of the range of $f(t)$. Hence, we can find a number $\delta > 0$ such that $[F(x_1), F(x_2)]_2 \leqq \varepsilon$ when $[x_1, x_2]_1 \leqq \delta$; $x_1, x_2 \in \mathfrak{A}$. It follows that the set $\{\tau_{F(f)}(\varepsilon)\}$ contains the set $\tau_f(\delta)$ and this completes the proof.

Theorem 3. *Let $\mathfrak{M}_1, \cdots, \mathfrak{M}_n$ be complete metric spaces and let $\mathfrak{M}$ denote their topological product. Let $f_\nu(t)$ for $\nu = 1, \cdots, n$ denote an almost periodic movement in $\mathfrak{M}_\nu$ and let $\mathfrak{A}$ denote the closure of the range of the almost periodic movement $\boldsymbol{f}(t) = (f_1(t), \cdots, f_n(t))$ in $\mathfrak{M}$. Let $y = F(x_1, \cdots, x_n)$ where y belongs to a metric space $\mathfrak{M}^*$, be a function continuous when $(x_1, \cdots,$*

$x_n) \in \mathfrak{A}$. *Then* $F(f_1(t), \cdots, f_n(f))$ *is an almost periodic movement in* $\mathfrak{M}^*$.

This follows immediately from Lemmas 8 and 10. The function $F(x_1, \cdots, x_n)$ may be considered as a composition rule in $\mathfrak{M}$. For example if we have some algebraic operation defined in $\mathfrak{M}$ and this operation is continuous, the set of all almost periodic movements in $\mathfrak{M}$ will be mapped into itself by this operation. If $\mathfrak{M}$ is a metric group, almost periodicity will be preserved by group multiplication.

The complex number sphere is metrized by the usual distance in the 3-dimensional Euclidean space and the ordinary addition and multiplication form an algebra on the sphere except at the point at infinity. At this point the operations are discontinuous. Therefore, it cannot be expected that addition or multiplication of almost periodic movements on the sphere will lead to almost periodic sums and products when the original movements get arbitrarily close to infinity. For more details on this question including explicit examples we refer to R. Norgil [11].

3. On the Spatial Extension of a Uniformly Continuous Family of Almost Periodic Movements in a Complete Metric Space.

We shall now study functions $x = G(u_1, u_2, \cdots)$, $-\infty < u_\nu < \infty$, $\nu = 1, 2, \cdots$; $x \in \mathfrak{M}$ depending on an infinite sequence of variables. We shall use the vectorial notation $\boldsymbol{u} = (u_1, u_2, \cdots)$, $\boldsymbol{u}' = (u_1', u_2', \cdots)$, $G(\boldsymbol{u}) = G(u_1, u_2, \cdots)$ and the linear operations $\boldsymbol{u}t = (u_1 t, u_2 t, \cdots)$, $\boldsymbol{u} + \boldsymbol{u}' = (u_1 + u_1', u_2 + u_2', \cdots)$. A neighbourhood of the vector $\boldsymbol{u}$ is defined as the set of vectors $\boldsymbol{u}'$ satisfying the inequalities

$$|u_\mu' - u_\mu| < \delta, \; \mu = 1, \cdots, m,$$

where m is a positive integer and $\delta > 0$.

Definition 6. *A function* $x = G(\boldsymbol{u})$, $-\infty < u_\nu < \infty$, $\nu = 1, 2, \cdots$, $x \in \mathfrak{M}$ *is called limit periodic with the limit period* 2π *in each variable if it satisfies the following condition: To* $\varepsilon > 0$

correspond $\delta > 0$ *and positive integers* m *and* N *such that*

$$[G(\boldsymbol{u}'), G(\boldsymbol{u}'')] \leqq \varepsilon$$

when

$$(3) \qquad |u''_\mu - u'_\mu| \leqq \delta \pmod{2N\pi}, \ \mu = 1, \cdots, m.$$

It follows immediately that we have the following lemma.

Lemma 11. *A function* $G(\boldsymbol{u})$ *with the limit period* 2π *in each variable is uniformly continuous,* i. e. *to* $\varepsilon > 0$ *correspond* $\delta > 0$ *and a positive integer* m *such that* $[G(\boldsymbol{u}'), G(\boldsymbol{u}'')] \leqq \varepsilon$, *when* $|u''_\mu - u'_\mu| \leqq \delta$, $\mu = 1, \cdots, m$.

The theorem follows immediately from Definition 10 since the conditions $|u''_\mu - u'_\mu| \leqq \delta$, $\mu = 1, \cdots, m$, imply (3).

Definition 7. *Let* $\mathfrak{K}$ *denote a compact S-space consisting of points denoted* $v, v_1, \cdots$ *and with neighbourhoods* $U(v)$. *A function* $x = G(\boldsymbol{u}; v) = G(u_1, u_2, \cdots; v)$, $-\infty < u_\nu < \infty$, $\nu = 1, 2, \cdots$; $v \in \mathfrak{K}$; $x \in \mathfrak{M}$, *is called a uniformly continuous family of limit periodic functions when*

1. *The function* $G(\boldsymbol{u}; v)$ *has the limit period* 2π *in each variable* u_μ *for every fixed* $v \in \mathfrak{K}$.

2. *To* $\varepsilon > 0$ *and* $v_0 \in \mathfrak{K}$ *corresponds a neighbourhood* $U_\varepsilon(v_0)$ *such that* $[G(\boldsymbol{u}; v_0), G(\boldsymbol{u}; v)] \leqq \varepsilon$ *when* $-\infty < u_\nu < \infty$, $\nu = 1, 2, \cdots$; $v \in U_\varepsilon(v_0)$.

Lemma 12. *A uniformly continuous family* $G(\boldsymbol{u}; v)$ *of limit periodic functions has the following property: To* $\varepsilon > 0$ *correspond* $\delta > 0$ *and positive integers* m *and* N *such that*

$$[G(\boldsymbol{u}'; v), G(\boldsymbol{u}''; v)] \leqq \varepsilon$$

when

$$|u''_\mu - u'_\mu| \leqq \delta \pmod{2N\pi}, \ \mu = 1, \cdots, m; \ v \in \mathfrak{K}.$$

Proof. We can choose $\delta(v_0) > 0$ and positive integers $m(v_0)$ and $N(v_0)$ such that

$$[G(\boldsymbol{u}'; v_0), G(\boldsymbol{u}''; v_0)] \leqq \frac{\varepsilon}{3}$$

when

$$|u''_\mu - u'_\mu| \leqq \delta(v_0) \pmod{2\pi N(v_0)}, \ \mu = 1, \cdots, m(v_0).$$

By condition 2 of Definition 7 it follows that

$$[G(\boldsymbol{u}'; v), G(\boldsymbol{u}''; v)] \leq \varepsilon$$

when

$$|u''_\mu - u'_\mu| \leq \delta(v_0) \pmod{2\pi N(v_0)}, \ \mu = 1, \cdots, m(v_0); \ v \,\epsilon\, U_{\frac{\varepsilon}{3}}(v_0).$$

As $\mathfrak{K}$ is a compact S-space, we may choose $v_1, \cdots, v_n \,\epsilon\, \mathfrak{K}$ such that $\mathfrak{K} \subset U_{\frac{\varepsilon}{3}}(v_1) \cup \cdots \cup U_{\frac{\varepsilon}{3}}(v_n)$. It follows that the statement in Lemma 12 holds if we choose

$$\delta = \operatorname{Min} \delta(v_\nu), \ m = \operatorname{Max} m(v_\nu), \ N = N(v_1) \cdots N(v_n).$$

This completes the proof.

A connection between uniformly continuous families of limit periodic functions and uniformly continuous families of almost periodic movements is indicated by the following theorem.

Theorem 4. *Let $G(\boldsymbol{u}; v)$ be a uniformly continuous family of limit periodic functions and let $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \cdots)$ be a constant real vector. The function $f(t; v) = G(\boldsymbol{\lambda} t; v) = G(\lambda_1 t, \lambda_2 t, \cdots; v)$ is then a uniformly continuous family of almost periodic movements.*

Proof. With the notations of Lemma 12 we have that a real number τ is a translation number of every function $f(t; v_0)$, $v_0 \,\epsilon\, \mathfrak{K}$, corresponding to $\varepsilon > 0$ if the inequalities

$$|\lambda_\mu \tau| \leq \delta \pmod{2N\pi}, \ \mu = 1, \cdots, m,$$

are satisfied. It follows from Bohl-Wennberg's theorem that these inequalities are satisfied for a relatively dense set of numbers τ.

Hence condition 1 of Definition 2 is satisfied. Condition 2 of Definition 2 is an immediate consequence of condition 2 of Definition 7. This completes the proof.

Lemma 13. *Let $x_1 = G_1(\boldsymbol{u})$, $x_2 = G_2(\boldsymbol{u})$; $x_1, x_2 \,\epsilon\, \mathfrak{M}$ be two functions with the limit period 2π and let $\boldsymbol{\beta} = (\beta_1, \beta_2, \cdots)$ be a real vector with rationally independent coordinates. If the almost periodic movements $G_1(\boldsymbol{\beta} t)$ and $G_2(\boldsymbol{\beta} t)$ are identical, the functions $G_1(\boldsymbol{u})$ and $G_2(\boldsymbol{u})$ are also identical.*

Proof. According to Definition 6 we can choose the numbers

δ_1, m_1, and N_1 corresponding to $G_1(\boldsymbol{u})$ and $\frac{\varepsilon}{2} > 0$ and similarly δ_2, m_2, and N_2 corresponding to $G_2(\boldsymbol{u})$ and $\frac{\varepsilon}{2}$. We put

$$\delta = \operatorname{Min}(\delta_1, \delta_2),\ m = \operatorname{Max}(m_1, m_2),\ N = N_1 N_2,$$

and it follows that we have

$$[G_1(\boldsymbol{u}),\ G_2(\boldsymbol{u})] \leq \varepsilon \tag{4}$$

when there exists a real number t such that

$$|\beta_\mu t - u_\mu| \leq \delta \pmod{2N\pi},\ \mu = 1, \cdots, m.$$

But it follows from KRONECKER's theorem that such a number t can be chosen for all possible values of $u_1, \cdots, u_m$. Hence (4) holds everywhere, and ε was an arbitrary positive number. Hence $[G_1(\boldsymbol{u}),\ G_2(\boldsymbol{u})] = 0$ for every $\boldsymbol{u}$ and this completes the proof.

Definition 8. *Let $x = G(\boldsymbol{u}; v)$ be a uniformly continuous family of limit periodic functions and let $\boldsymbol{\beta} = (\beta_1, \beta_2, \cdots)$ be a real vector with rationally independent coordinates. The uniformly continuous family $x = f(t; v) = G(\boldsymbol{\beta} t; v)$ of almost periodic movements is called the diagonal family of $G(\boldsymbol{u}; v)$ corresponding to $\boldsymbol{\beta}$ and the family $x = G(\boldsymbol{u}; v)$ is called the spatial extension of $f(t; v)$ corresponding to $\boldsymbol{\beta}$.*

It follows from Lemma 13 that the spatial extension of a uniformly continuous family of almost periodic movements $x = f(t; v)$ corresponding to a vector $\boldsymbol{\beta}$ is uniquely determined by $f(t; v)$ and $\boldsymbol{\beta}$. On the other hand it does not always exist, but we shall prove that we can choose the vector $\boldsymbol{\beta}$ such that a spatial extension of $f(t; v)$ corresponding to $\boldsymbol{\beta}$ exists. For the proof we need the notion of a rational basis of a uniformly continuous family of almost periodic movements.

Definition 9. *A sequence $\beta_1, \beta_2, \cdots$ of rationally independent numbers is called a basis of $x = f(t; v)$ if the following condition is satisfied: To $\varepsilon > 0$ correspond $\eta > 0$ and positive integers m and N such that every real number τ satisfying the conditions*

$$|\beta_\mu \tau| \leq \eta \pmod{2N\pi},\ \mu = 1, \cdots, m \tag{5}$$

is a common translation number corresponding to ε of all almost periodic functions of the family $x = f(t; v)$.

We shall first prove the existence of a basis.

Theorem 5. *Every uniformly continuous family of almost periodic movements in $\mathfrak{M}$ has a basis.*

Proof. Let $f(t; v)$ be the given family. To every $\varepsilon > 0$ corresponds, according to Theorem 2, a $(\delta; \lambda)$-neighbourhood of zero such that every real number in this neighbourhood is a common ε-translation number of the functions $f(t; v)$. Let us consider the particular case where $\varepsilon = \frac{1}{n}$, $n = 1, 2, \cdots$. We arrange all numbers λ occurring in the definitions of the corresponding $(\delta; \lambda)$-neighbourhoods in a single sequence $\lambda_1, \lambda_2, \cdots$ and we may then write the statement of Theorem 2 on the form: To $\varepsilon > 0$ corresponds $\delta > 0$ and an integer m such that every number τ satisfying

$$(6) \qquad |\lambda_\mu \tau| \leqq \delta \pmod{2\pi}, \ \mu = 1, \cdots, m$$

is a common ε-translation number.

According to a well-known theorem we can find a sequence $\beta_1, \beta_2, \cdots$ of rationally independent numbers such that every λ_μ has a representation

$$\lambda_\mu = r_{\mu 1}\beta_1 + \cdots + r_{\mu q_\mu}\beta_{q_\mu}$$

where the numbers $r_{\mu\nu}$ are rational. Let N denote the common denominator of the numbers $r_{\mu\nu}$, $\mu \leqq m$ and let η be chosen such that

$$0 < \eta < \underset{1 \leqq \mu \leqq m}{\mathrm{Min}} \frac{\delta}{|r_{\mu 1}| + \cdots + |r_{\mu q_\mu}|}.$$

With this choice of the constants the inequalities (6) follow from (5). This proves that the sequence $\beta_1, \beta_2, \cdots$ is a basis of $f(t; v)$.

Theorem 6. *If the sequence $\beta_1, \beta_2, \cdots$ is a basis of a uniformly continuous family of almost periodic movements $f(t; v)$, there exists a spatial extension of $f(t; v)$ corresponding to $\beta = \beta_1, \beta_2, \cdots$.*

Proof. A similar proof for ordinary almost periodic functions was given by E. Pedersen [12]. Let $u_1, u_2, \cdots$ be a given sequence of real numbers. If n is a positive integer, it follows from KRONECKER's theorem that we can find a real number $t_n(\boldsymbol{u})$ such that

$$|\beta_\nu t_n(\boldsymbol{u}) - u_\nu| \leq \frac{1}{n} \pmod{n!\,2\pi}, \quad \nu = 1, \cdots, n.$$

If p is a positive integer, it follows that

$$|\beta_\nu (t_{n+p}(\boldsymbol{u}) - t_n(\boldsymbol{u}))| \leq \frac{2}{n} \pmod{n!\,2\pi}, \quad \nu = 1, \cdots, n.$$

By comparison with the conditions (5) we find that $t_{n+p}(\boldsymbol{u}) - t_n(\boldsymbol{u})$ is a common translation number of $f(t; v)$ corresponding to $\varepsilon > 0$ if n is large enough. Since $\mathfrak{M}$ is complete, it follows that there exists a limit function

$$G(\boldsymbol{u}; v) = \lim_{n\to\infty} f(t_n(\boldsymbol{u}); v).$$

The function $G(\boldsymbol{u}; v)$ is independent of the particular choice of the sequence of functions $t_n(\boldsymbol{u})$. It is not very difficult to prove this, but we find it more convenient to prove first that $G(\boldsymbol{u}; v)$ is a uniformly continuous family of limit periodic functions.

Let $\varepsilon > 0$ be given. According to Definition 9 we can find a corresponding $\eta > 0$ and positive integers m and N such that every real τ satisfying (5) is a common ε-translation number. If $\boldsymbol{u}'$ and $\boldsymbol{u}''$ are two vectors satisfying

$$(7) \qquad |u''_\nu - u'_\nu| \leq \frac{\eta}{3} \pmod{2N\pi}, \quad \nu = 1, \cdots, m,$$

we have for $n \geq \operatorname{Max}(m, N)$ that the following inequalities hold mod $2N\pi$ when $\nu = 1, \cdots, m$.

$$|\beta_\nu (t_n(\boldsymbol{u}'') - t_n(\boldsymbol{u}'))| \leq$$
$$|\beta_\nu t_n(\boldsymbol{u}'') - u''_\nu| + |u''_\nu - u'_\nu| + |u'_\nu - \beta_\nu t_n(\boldsymbol{u}')| \leq \frac{\eta}{3} + \frac{2}{n},$$

and if we further choose $n \geq \frac{6}{\eta}$, we obtain

$$|\beta_\nu (t_n(\boldsymbol{u}'') - t_n(\boldsymbol{u}'))| \leq \eta \pmod{2N\pi}, \quad \nu = 1, \cdots, m.$$

Thus, the inequalities (5) are satisfied by $t_n(\boldsymbol{u}'') - t_n(\boldsymbol{u}')$ and we have

$$[f(t_n(\boldsymbol{u}'); v),\, f(t_n(\boldsymbol{u}''); v)] \leqq \varepsilon.$$

It follows that

$$(8) \qquad [G(\boldsymbol{u}'; v),\, G(\boldsymbol{u}''; v)] \leqq \varepsilon$$

when (7) is satisfied and this proves that the first condition of Definition 7 is satisfied. If $v_0 \in \mathfrak{K}$ is given, it follows from Definition 2 that we can find a neighbourhood $U_\varepsilon(v_0)$ such that

$$[f(t; v_0),\, f(t; v)] \leqq \varepsilon \text{ when } -\infty < t < \infty,\ v \in U_\varepsilon(v_0).$$

We have then in particular

$$[f(t_n(\boldsymbol{u}); v_0),\, f(t_n(\boldsymbol{u}); v)] \leqq \varepsilon$$

for all $\boldsymbol{u}$ and n when $v \in U_\varepsilon(v_0)$, hence for $n \to \infty$

$$[G(\boldsymbol{u}; v_0),\, G(\boldsymbol{u}; v)] \leqq \varepsilon,\ v \in U_\varepsilon(v_0).$$

This proves that the second condition of Definition 7 is satisfied and the function $G(\boldsymbol{u}; v)$ is a uniformly continuous family of limit periodic functions.

This implies that $G(\boldsymbol{u}; v)$ is independent of the choice of the sequence $t_n(\boldsymbol{u})$. In fact, if we made another choice of the sequence $t_n(\boldsymbol{u}^*)$ for just one fixed $\boldsymbol{u}^*$ and thereby changed the value of $G(\boldsymbol{u}^*; v^*)$, this would certainly disturb the continuity of the limit periodic function $G(\boldsymbol{u}; v^*)$.

If, in particular, $\boldsymbol{u} = \boldsymbol{\beta} t$, we may always choose $t_n(\boldsymbol{u}) = t$ and we obtain $G(\boldsymbol{\beta} t; v) = f(t; v)$. This completes the proof of Theorem 6.

In some special cases the preceding result may be simplified. We shall first consider a function with a finite basis.

Definition 10. *If the number m in Definition 9 can be chosen independent of ε, the finite sequence $\beta_1, \cdots, \beta_m$ is called a finite basis of $f(t; v)$.*

In this case the numbers $\beta_{m+1}, \beta_{m+2}, \cdots$ may obviously be chosen at random, since they do not occur in the conditions (5).

Theorem 7. *If $\beta_1, \cdots, \beta_m$ in Theorem 6 is a finite basis of $f(t; v)$, the spatial extension $G(\boldsymbol{u}; v)$ is independent of u_{m+1}, $u_{m+2}, \cdots$.*

Proof. If the vectors $\boldsymbol{u}'$ and $\boldsymbol{u}''$ satisfy the condition $u'_\nu = u''_\nu$, $\nu = 1, \cdots, m$, the condition (7) is satisfied for every $\eta > 0$ and every positive integer N. Hence (8) is satisfied for every $\varepsilon > 0$. This completes the proof. In this case we write $\boldsymbol{\beta} = (\beta_1, \cdots, \beta_m)$ and $G(\boldsymbol{u}; v) = G(u_1, \cdots, u_m; v)$.

Definition 11. *The basis $\beta_1, \beta_2, \cdots$ is called integral if we can always choose $N = 1$ in Definition 7.*

This happens if the coefficients $r_{\mu\nu}$ in the proof of Theorem 5 are integers.

Theorem 8. *If the basis $\beta_1, \beta_2, \cdots$ in Theorem 6 is integral, the spatial extension has the period 2π in each variable.*

Proof. When $N = 1$, the conditions (7) are obviously satisfied when

$$u''_\nu = u'_\nu + 2 h_\nu \pi, \; \nu = 1, 2, \cdots$$

where the numbers h_ν are integers. With this connection between $\boldsymbol{u}'$ and $\boldsymbol{u}''$ the relation (8) is then satisfied for every $\varepsilon > 0$. This completes the proof.

Finally, we shall prove the following lemmas on families of limit periodic functions.

Lemma 14. *To a uniformly continuous family $G(\boldsymbol{u}; v)$ of limit periodic functions and a given $\varepsilon > 0$ corresponds a uniformly continuous family $G^*(\boldsymbol{u}; v) = G^*(u_1, \cdots, u_m; v)$ of limit periodic functions depending only on a finite number of variables and satisfying*

$$[G^*(\boldsymbol{u}; v), G(\boldsymbol{u}; v)] \leq \varepsilon.$$

Proof. From Lemma 12 follows immediately that

$$[G(u_1, \cdots, u_m, 0, 0, \cdots; v), G(u_1, u_2, \cdots; v)] \leq \varepsilon,$$

which proves the lemma.

Lemma 15. *The closure of the range of a uniformly continuous family of limit periodic functions is a compact set.*

Proof. With the notations of Lemma 12 the range of the function

$$x = G(u_1, \cdots, u_m, 0, 0, \cdots; v), 0 \leqq u_\mu \leqq 2N\pi, \mu = 1, \cdots, m; v \varepsilon K$$

is a compact set, which comes within ε-distance of every point of the closure of the range of $G(\boldsymbol{u}; v)$. As ε is arbitrary, this implies Lemma 15.

4. The Approximation Theorem.

It will now be necessary to restrict the investigations to a much more special type of metric spaces. The conditions which we are going to impose upon the space $\mathfrak{M}$ are stronger than necessary for the validity of the approximation theorem, but they are rather simple and the proof will not be too difficult.

Definition 12. *The metric space $\mathfrak{M}$ is called continuously locally arcwise connected when it satisfies the following condition: To every compact subset $\mathfrak{C}$ of $\mathfrak{M}$ corresponds a positive number Δ and a continuous function $z = \varphi(x; t; y)$, $0 \leqq t \leqq 1$, $x, y \,\varepsilon\, \mathfrak{C}$, $[x, y] \leqq \Delta$; $z \,\varepsilon\, \mathfrak{M}$ satisfying the conditions*

$$\varphi(x; 0; y) = x, \ \varphi(x; 1; y) = y; \ \varphi(x; t; x) = x.$$

This condition is satisfied if any two points x and y of $\mathfrak{M}$ whose distance remains below a certain number can be connected by a geodetic arc which depends continuously on x and y.

Lemma 16. *Let $\mathfrak{C}$ be a compact subset of $\mathfrak{M}$. To $\varepsilon > 0$ corresponds $\delta > 0$ such that $[x, \varphi(x; t; y)] \leqq \varepsilon$ when $0 \leqq t \leqq 1$, $x, y \,\varepsilon\, \mathfrak{C}$, $[x, y] \leqq \delta$.*

Proof. As $\varphi(x; t; y)$ is uniformly continuous when $0 \leqq t \leqq 1$, $x, y \,\varepsilon\, \mathfrak{C}$, we can determine δ such that $[\varphi(x; t; x), \varphi(x; t; y)] \leqq \varepsilon$ when $0 \leqq t \leqq 1$, $x, y \,\varepsilon\, \mathfrak{C}$, $[x, y] \leqq \delta$. Since $x = \varphi(x; t; x)$, this proves the lemma.

We observe that the conditions in Definition 12 do not imply that $\mathfrak{M}$ is complete. In fact, an open segment of a straight line with the ordinary metric satisfies the conditions, but it is not complete.

We are going to prove that a uniformly continuous family of limit periodic functions can be approximated with any given accuracy by another family depending only on a finite number of variables and continuous and periodic in each of these. This is an elementary consequence of the conditions in Definition 12, but the proof presents some technical difficulties. To make the proof more perspicuous we shall make use of some notions introduced by the following definition.

Definition 13. *Let $G(\boldsymbol{u}; v)$ be a uniformly continuous family of limit periodic functions with values from $\mathfrak{M}$. If we give the variable u_ν a fixed value a, we obtain a function, which we shall denote $G_{\nu; a}(\boldsymbol{u}; v)$. Let N be a positive integer and d a positive number such that* $[G(\boldsymbol{u}'; v), G(\boldsymbol{u}''; v)] \leq \Delta$ (Definition 12), *when* $|u_\nu'' - u_\nu'| \leq d \pmod{2N\pi}$ *and* $u_\mu' = u_\mu''$, $\mu \neq \nu$. *We define $H(\boldsymbol{u}; v) = T_{\nu; N, d}\, G(\boldsymbol{u}; v)$ in the following way.*

$$H(\boldsymbol{u}; v) = G(\boldsymbol{u}; v) \text{ when } 0 \leq u_\nu \leq 2N\pi - d.$$

$$H(\boldsymbol{u}; v) = \varphi\left(G_{\nu; 2N\pi - d}(\boldsymbol{u}; v); \frac{u_\nu - 2N\pi + d}{d}; G_{\nu; 0}(\boldsymbol{u}; v)\right)$$
$$\text{when } 2N\pi - d \leq u_\nu \leq 2N\pi.$$

$H(\boldsymbol{u}; v)$ is periodic in u_ν with the period $2N\pi$.

The function $H(\boldsymbol{u}; v) = T_{\nu; N, d}\, G(\boldsymbol{u}; v)$ is called a periodification of $G(\boldsymbol{u}; v)$ with respect to the variable u_ν.

We observe that it follows from the properties of $\varphi(x; t; y)$ that the definition of $H(\boldsymbol{u}; v)$ is not ambiguous when $u_\nu = 2N\pi - d$ or when u_ν is a multiple of $2N\pi$. We shall first prove the following lemma.

Lemma 17. *The periodification $H(\boldsymbol{u}; v)$ is a uniformly continuous family of limit periodic functions.*

According to Lemma 15 there exists a compact set $\mathfrak{C} \subset \mathfrak{M}$ which contains all values assumed by $G(\boldsymbol{u}; v)$. The function $\varphi(x; t; y)$ is uniformly continuous when $0 \leq t \leq 1$, $x, y \in \mathfrak{C}$, $[x, y] \geq \Delta$. Let $\varepsilon > 0$ be given. We can then choose two positive numbers η_1 and η_2 such that

$$[\varphi(x; t_1; y), \varphi(x; t_2; y)] \leq \frac{\varepsilon}{4} \text{ when } |t_2 - t_1| \leq \eta_1$$

and

$$[\varphi(x_1; t; y_1), \varphi(x_2; t; y_2)] \leq \frac{\varepsilon}{2} \text{ when } [x_1, x_2] \leq \eta_2,\ [y_1, y_2] \leq \eta_2.$$

In both cases we must, of course, assume that each x and y belongs to $\mathfrak{E}$ and that every t belongs to the interval $0 \leqq t \leqq 1$.

We choose a positive number $\delta_1 \leqq \eta_1 d$ such that

$$[G(\boldsymbol{u}'; v), G(\boldsymbol{u}''; v)] \leqq \frac{\varepsilon}{4} \text{ when } |u''_\nu - u'_\nu| \leqq \delta_1;\ u'_\mu = u''_\mu,\ \mu \neq \nu;\ v \in \mathfrak{K}$$

and it follows immediately that

$$[H(\boldsymbol{u}'; v), H(\boldsymbol{u}''; v)] \leqq \frac{\varepsilon}{2}$$

when

$$|u''_\nu - u'_\nu| \leqq \delta_1 \pmod{2N\pi};\ u'_\mu = u''_\mu,\ \mu \neq \nu;\ v \in \mathfrak{K}.$$

We choose a positive number δ_2 and positive integers N^* and $m \geqq \nu$ such that

$$[G(\boldsymbol{u}'; v), G(\boldsymbol{u}''; v)] \leqq \operatorname{Min}\left(\frac{\varepsilon}{2}, \eta_2\right)$$

when

$$|u''_\mu - u'_\mu| \leqq \delta_2 \pmod{2N^*\pi},\ \mu = 1, \cdots, m;\ v \in \mathfrak{K}.$$

It follows immediately that

$$[H(\boldsymbol{u}'; v), H(\boldsymbol{u}''; v)] \leqq \frac{\varepsilon}{2}$$

when

$$u'_\nu = u''_\nu;\ |u''_\mu - u'_\mu| \leqq \delta_2 \pmod{2N^*\pi},\ \mu = 1, \cdots, m;\ v \in \mathfrak{K}.$$

If we combine our two results, we obtain

$$[H(\boldsymbol{u}'; v), H(\boldsymbol{u}''; v)] \leqq \varepsilon$$

when

$$|u''_\mu - u'_\mu| \leqq \operatorname{Min}(\delta_1, \delta_2) \pmod{2NN^*\pi},\ \mu = 1, \cdots, m;\ v \in \mathfrak{K}.$$

Let $v_0 \in \mathfrak{K}$ be given. We choose a neighbourhood $U(v_0)$ such that $[G(\boldsymbol{u}; v_0), G(\boldsymbol{u}; v)] \leqq \operatorname{Min}(\varepsilon, \eta_2)$ when $v \in U(v_0)$. It follows that

$$[H(\boldsymbol{u}; v_0), H(\boldsymbol{u}; v)] \leqq \varepsilon \text{ when } v \in U(v_0).$$

This completes the proof of Lemma 17.

Lemma 18. *If the family $G(\boldsymbol{u}; v)$ of Definition* 13 *is periodic with the period* $2N_1\pi$ *in the variable* u_μ, *the function* $H(\boldsymbol{u}; v)$ *also has the period* $2N_1\pi$ *in the variable* u_μ. *If* $G(\boldsymbol{u}; v)$ *is independent of* u_μ, *the function* $H(\boldsymbol{u}; v)$ *is also independent of* u_μ.

Proof. The theorem follows immediately from the definition of $H(\boldsymbol{u}; v)$.

Lemma 19. *Let* $\varepsilon > 0$ *be given. We can choose* $d > 0$ *and a positive integer* N *such that the family* $H(\boldsymbol{u}; v)$ *of Definition* 13 *satisfies the condition* $[H(\boldsymbol{u}; v),\ G(\boldsymbol{u}; v)] \leq \varepsilon$ *for every vector* $\boldsymbol{u}$ *and every* $v \in \mathfrak{R}$.

Proof. According to Lemmas 15 and 16 we can choose $\eta > 0$ such that $[x, \varphi(x; t; y)] < \frac{\varepsilon}{2}$ when $0 \leq t \leq 1$ and x and y are values assumed by $G(\boldsymbol{u}; v)$ and satisfying $[x, y] \leq \eta$. We can then, according to Lemma 12, choose $d \geq 0$ and a positive integer N such that

$$[G(\boldsymbol{u}'; v),\ G(\boldsymbol{u}''; v)] \leq \operatorname{Min}\left(\frac{\varepsilon}{2}, \eta\right)$$

when

$$u'_\mu = u''_\mu,\ \mu \neq \nu;\ \left| u''_\nu - u'_\nu \right| \leq d \ (\mathrm{mod}\ 2N\pi);\ v \in \mathfrak{R}.$$

Let $\boldsymbol{u}$ be arbitrary. There exists a vector $\boldsymbol{u}^*$ and an integer q satisfying

$$u^*_\mu = u_\mu,\ \mu \neq \nu;\ u_\nu - u^*_\nu = 2qN\pi;\ 0 \leq u^*_\nu \leq 2N\pi.$$

We have

$$H(\boldsymbol{u}; v) = H(\boldsymbol{u}^*; v);\ [G(\boldsymbol{u}; v),\ G(\boldsymbol{u}^*; v)] \leq \frac{\varepsilon}{2}.$$

If $0 \leq u^*_\nu \leq 2N\pi - d$, we have $H(\boldsymbol{u}^*; v) = G(\boldsymbol{u}^*; v)$, hence

$$[H(\boldsymbol{u}; v),\ G(\boldsymbol{u}; v)] \leq \frac{\varepsilon}{2}.$$

If $2N\pi - d < u^*_\nu \leq 2N\pi$, we have

$$[G_{\nu; 2N\pi - d}(\boldsymbol{u}; v),\ G(\boldsymbol{u}; v)] \leq \frac{\varepsilon}{2}$$

and

$$[G_{\nu;2N\pi-d}(\boldsymbol{u};v),\ H(\boldsymbol{u};v)] =$$
$$\left[G_{\nu;2N\pi-d}(\boldsymbol{u};v),\ \varphi\left(G_{\nu;2N\pi-d}(\boldsymbol{u};v);\ \frac{u_\nu^*-2N\pi+d}{d};\ G_{\nu;0}(\boldsymbol{u};v)\right)\right] \leq \frac{\varepsilon}{2}$$

since $[G_{\nu;2N\pi-d}(\boldsymbol{u};\ v),\ G_{\nu;0}(\boldsymbol{u};\ v] \leq \eta$. Hence, we have in this case $[H(\boldsymbol{u};\ v),\ G(\boldsymbol{u};\ v)] \leq \varepsilon$ and this completes the proof.

We can now prove the approximation theorem for uniformly continuous families of limit periodic functions.

Theorem 9. *Let $G(\boldsymbol{u};v)$ be a uniformly continuous family of limit periodic functions with values from a complete and continuously locally arcwise connected space $\mathfrak{M}$ and let ε be a positive number. There exist two positive integers m and N and a continuous function $g(\boldsymbol{u};v) = g(u_1,\cdots,u_m;v)$ with the period $2N\pi$ in each variable, satisfying $[g(\boldsymbol{u};v),\ G(\boldsymbol{u};v)] \leq \varepsilon$ for every vector $\boldsymbol{u}$ and every $v \in \mathfrak{K}$.*

Proof. From Lemma 14 follows that we can find an integer m and a uniformly continuous family $g_0(\boldsymbol{u};v) = g_0(u_1,\cdots,u_m;v)$ of limit periodic functions satisfying

$$[g_0(\boldsymbol{u};v),\ G(\boldsymbol{u};v)] \leq \frac{\varepsilon}{2}.$$

To prove the theorem we need only construct a finite sequence $g_0(\boldsymbol{u};v),\ \cdots,\ g_m(\boldsymbol{u};v)$ of uniformly continuous families of limit periodic functions satisfying the conditions

1). $[g_\nu(\boldsymbol{u};v),\ g_{\nu-1}(\boldsymbol{u};v)] \leq \dfrac{\varepsilon}{2m},\ \nu = 1,\cdots,m.$

2). $g_\nu(\boldsymbol{u};v)$ is independent of the variables $u_{m+1}, u_{m+2},\cdots$.

3). There exists a sequence $N_1,\cdots,N_m$ of positive integers such that $g_\nu(\boldsymbol{u};v)$ has the periods $2N_1\pi,\cdots,2N_\nu\pi$ in the variables $u_1,\cdots,u_\nu$.

In fact, $g_m(\boldsymbol{u};v)$ depends only on $u_1,\cdots,u_m$, it has the period $2N_1\cdots N_m\pi$ in each of these variables, and it satisfies the condition

$$[g_m(\boldsymbol{u};v),\ G(\boldsymbol{u};v] \leq$$
$$\sum_{\nu=1}^{m}[g_\nu(\boldsymbol{u};v),\ g_{\nu-1}(\boldsymbol{u};v)] + [g_0(\boldsymbol{u};v),\ G(\boldsymbol{u};v)] \leq m\frac{\varepsilon}{2m} + \frac{\varepsilon}{2} = \varepsilon.$$

Let us now assume that we have already constructed the functions $g_0(\boldsymbol{u}; v), \cdots, g_{\nu-1}(\boldsymbol{u}; v)$ satisfying 1)—3). According to Lemma 19 we can choose a periodification $g_\nu(\boldsymbol{u}; v) = T_{\nu; N_\nu, d}\, g_{\nu-1}(\boldsymbol{u}; v)$ such that 1) is satisfied. From Lemma 17 follows that $g_\nu(\boldsymbol{u}; v)$ is a uniformly continuous family of limit periodic functions, and from Lemma 18 follows that the conditions 2) and 3) are satisfied. This completes the proof.

If we now combine Theorems 5, 6, and 9, we obtain the following approximation theorem.

Theorem 10. *Let $\mathfrak{M}$ be a complete, continuously locally arcwise connected space. To a uniformly continuous family of almost periodic movements $x = f(t; v)$, $-\infty < t < \infty$, $v \in \mathfrak{K}$; $x \in \mathfrak{M}$ and a positive number ε correspond positive integers m and N, a continuous function $g(u_1, \cdots, u_m; v)$, $-\infty < u_\nu < \infty$, $\nu = 1, \cdots, m$; $v \in \mathfrak{K}$ with the period $2N\pi$ in each of the variables u_ν, and rationally independent numbers $\beta_1, \cdots, \beta_m$ such that*

$$[g(\beta_1 t, \cdots, \beta_m t; v), f(t; v)] \leq \varepsilon, \; -\infty < t < \infty, \; v \in \mathfrak{K}.$$

We could have proved this theorem by a more direct method similar to a variant of Bogolioùboff's proof of the theorem for ordinary almost periodic functions (Cf. [10] p. 96—109). By this method the existence of the spatial extension is not proved and it is not necessary to assume that $\mathfrak{M}$ is complete. We have preferred the longer proof because it yields the theorem on the existence of the spatial extension more directly.

5. A Complete Metric Space, in which the Approximation Theorem for Almost Periodic Movements does not hold.

We have seen that the approximation theorem for almost periodic movements holds in every complete, continuously locally arcwise connected space. Although this property of the space is not necessary, we shall prove that it is not superfluous. To do this, we find a complete metric space in which the approximation theorem does not hold. We shall use one of the so-called solenoidal spaces, introduced and investigated by D. van Dantzig [6]. He uses the following construction.

In a half-plane p bounded by a straight line l a circular disk c_1 is situated at a positive distance h from l. A circular disk c_2 has its center at a fixed point of c_1, a circular disk c_3 has its center at a fixed point of c_2, and so on. The boundary of c_{n+1} shall belong to the interior of c_n and the center of c_n shall be outside c_{n+1}. Let $q_1, q_2, \cdots$ be a sequence of integers greater than 1. For every n the disk c_n with the center of c_{n+1} rigidly attached to it revolves in its plane around its center with the angular velocity $(q_1 \cdots q_n)^{-1}$, while the plane p rotates about the line l with the angular velocity 1. The boundary of the disk c_n then describes a closed tube T_n, which is wound $q_1 \cdots q_n$ times around l. The radius of c_n is chosen so small that the $q_1 \cdots q_n$ strands of T_n are entirely separate. The intersection of all the tubes T_n is called a solenoid. It is a perfect point set in 3-dimensional space. Considered as a relative space the solenoid is complete, compact, and connected.

At every fixed moment during the rotation, the half-plane p intersects the interior of the tube T_n in $q_1 \cdots q_n$ circles congruent to c_n. For every n we choose one of these circles c_n^* such that $c_1 = c_1^* \supset c_2^* \supset c_3^* \supset \cdots$. Exactly one point x of the half-plane p will be contained in all these circles. When p rotates, the circles of intersection c_n^* move continuously in the plane and the point x defined by them performs a continuous movement $x = g(t)$ in the solenoid. This movement is not periodic, but the movements performed by the centers of $c_1^*, c_2^*, \cdots$ are periodic and converge uniformly to $g(t)$. Hence, $g(t)$ is an almost periodic movement in the solenoid. The center of c_n^* does not belong to the solenoid, and we shall see that $g(t)$ cannot be approximated by any diagonal movement in the solenoid with an accuracy better than $2h$.

The curve $x = g(t)$ has no double points. Van Dantzig called it a pseudo-component of the solenoid, and he proved that two points of two differents pseudo-components cannot be connected by a continuous curve, contained in the solenoid ([6], p. 116). Let $x = f(\boldsymbol{u}) = f(u_1, \cdots, u_m)$, $-\infty < u_\nu < \infty$, $\nu = 1, \cdots, m$, where x is a variable point of the solenoid, be a continuous function. It follows that x belongs to one pseudo-component of the solenoid, and we can choose one of the points x_0, in which this pseudo-component intersects p at the time

$t = 0$. The position of x is then determined by the angle θ of rotation of p, which makes x_0 slide along the pseudo-component until it reaches the position of x. Thus, the function $x = f(\boldsymbol{u})$ determines a continuous function $\theta = \theta(\boldsymbol{u})$. If $f(\boldsymbol{u})$ has the period 2π in each variable, the function $\theta(\boldsymbol{u})$ also has the period 2π. To the diagonal function $f(\boldsymbol{\beta} t)$, where $\beta_1 \cdots, \beta_m$ are rationally independent numbers, corresponds the bounded function $\theta = \theta(\boldsymbol{\beta} t)$. The movement $g(t)$ introduced above corresponds to the function $\theta_1 = t$. For some value of t we have $\theta_1 - \theta = \pi$ and for this particular value the distance between $f(\boldsymbol{\beta} t)$ and $g(t)$ is $> 2h$. This proves that the approximation theorem for almost periodic movements does not hold in the solenoid.

Chapter 2.

Homotopic Almost Periodic Movements.

1. Preliminary Definitions and Investigations.

In the present chapter we shall study some simple questions in connection with the topology of almost periodic movements in a metric space $\mathfrak{M}$. From a topological point of view two almost periodic movements are considered essentially identical to one another if one of them can be changed in a uniformly continuous way into the other. It must be specified, however, whether or not the almost periodicity shall be preserved during the process of changing one movement into the other. If we require that the almost periodicity shall be preserved, we get a finer classification and we prefer this point of view. Hence, the following definition.

Definition 14. *Two almost periodic movements* $x_1 = f_1(t)$, $x_2 = f_2(t)$, $-\infty < t < \infty$; $x_1, x_2 \in \mathfrak{M}$ *are called (almost periodically) homotopic to one another if there exists a uniformly continuous family* $x = f(t; v)$, $-\infty < t < \infty$, v *real,* $\alpha \leqq v \leqq \beta$, $x \in \mathfrak{M}$ *satisfying* $f(t; \alpha) = f_1(t)$, $f(t; \beta) = f_2(t)$.

The relation of almost periodic homotopy is an equivalence relation in the set of almost periodic movements in $\mathfrak{M}$ and it leads to a division of this set into almost periodic homotopy classes. A first object of a topology of almost periodic movements will be a method to attach to every almost periodic movement certain constants characteristic for the class of the movement. In the present paper we shall solve only a small fraction of this problem. We shall find a property characteristic for the metric spaces in which every homotopy class contains a periodic movement. We shall always assume that the space $\mathfrak{M}$ is complete and continuously locally arcwise connected.

Lemma 20. *Every almost periodic movement* $x_1 = f_1(t)$ *in* $\mathfrak{M}$ *is homotopic to a certain diagonal movement* $x_2 = g(\boldsymbol{\beta} t) = g(\beta_1 t, \cdots, \beta_m t)$ *where* $g(\boldsymbol{u})$ *has the period* 2π *in each variable and* $\beta_1, \cdots, \beta_m$ *are rationally independent numbers.*

Proof. Let Δ be the positive number introduced in Definition 12. According to Theorem 10 we can choose $g(\boldsymbol{u})$ and $\boldsymbol{\beta} = (\beta_1, \cdots, \beta_m)$ such that $[g(\boldsymbol{\beta} t), f_1(t)] \leqq \Delta$, $-\infty < t < \infty$. We define

$$f(t; v) = \varphi(f_1(t); v; g(\boldsymbol{\beta} t)), \; 0 \leqq v \leqq 1$$

where φ is the function introduced in Definition 16. Let $\varepsilon > 0$ be given. As $f_1(t)$ and $g(\beta_1 t, \cdots, \beta_m t)$, according to Lemma 9, are contained in a compact subset of $\mathfrak{M}$, we can find $\delta > 0$ such that

$$[f(t; v), f(t+\tau; v)] \leqq \varepsilon$$

when τ is a common translation number of $f_1(t)$ and $g(\boldsymbol{\beta} t)$ corresponding to δ. However, it follows from Lemma 8 that the set of these numbers is relatively dense. Hence, the first condition in Definition 2 is satisfied. The second condition follows immediately from the uniform continuity of φ. From Definition 12 follows finally

$$f(t; 0) = f_1(t), \; f(t; 1) = g(\boldsymbol{\beta} t).$$

This completes the proof.

Remark. *The movement* $x_2 = g(\boldsymbol{\beta} t)$ *in Lemma* 20 *is just an arbitrary diagonal function, which approximates* $f_1(t)$ *with the accuracy* Δ.

2. On Reducible Almost Periodic Movements.

We shall prove the following lemma.

Lemma 21. *Let* $g(\boldsymbol{u}) = g(u_1, \cdots, u_m)$ *be a continuous function with the period* 2π *in each variable and let* $\beta_1, \cdots, \beta_m$ *be* *rationally independent numbers. The diagonal movement* $f(t) = g(\boldsymbol{\beta} t) = g(\beta_1 t, \cdots, \beta_m t)$ *is periodic if and only if* $g(\boldsymbol{u})$ *has the form* $h(n_1 u_1 + \cdots + n_m u_m)$ *where* $n_1, \cdots, n_m$ *are integers and* $h(u)$ *has the period* 2π.

Proof. The function $g(n_1 \beta_1 t + \cdots + n_m \beta_m t)$ is obviously periodic with the period $2\pi(n_1\beta_1 + \cdots + n_m\beta_m)^{-1}$. This proves the sufficiency. On the other hand, if $f(t)$ has the period p, a vector $\boldsymbol{\xi} = (\xi_1, \cdots, \xi_m)$ is a period vector of $g(\boldsymbol{u})$ if the inequalities

$$|\beta_\mu p n - \xi_\mu| \leq \varepsilon \pmod{2\pi}, \quad \mu = 1, \cdots, m$$

can be solved with respect to n for every $\varepsilon > 0$. This condition is satisfied if and only if the inequalities

$$(9) \quad \left\{ \begin{array}{l} |\beta_\mu p \tau - \xi_\mu| \leq \varepsilon \pmod{2\pi}, \quad \mu = 1, \cdots, m, \\ \qquad\qquad |2\pi\tau| \leq \varepsilon \pmod{2\pi} \end{array} \right.$$

can be solved with respect to τ for every $\varepsilon > 0$.

If the numbers $\beta_1, \cdots, \beta_m, \dfrac{2\pi}{p}$ are rationally independent, the inequalities (9) have solutions for every vector $\boldsymbol{\xi}$ and every $\varepsilon > 0$, according to Kronecker's theorem; but this implies that $g(\boldsymbol{u})$ is constant.

If $\beta_1, \cdots, \beta_m, \dfrac{2\pi}{p}$ are not rationally independent, they satisfy a relation

$$(10) \qquad N \cdot \frac{2\pi}{p} = n_1\beta_1 + \cdots + n_m\beta_m$$

where $N, n_1, \cdots, n_m$ are integers, some of which are different from zero. All possible linear relations with rational coefficients satisfied by the numbers $\beta_1, \cdots, \beta_m, \dfrac{2\pi}{p}$, are obtained from (10) by multiplication with an arbitrary rational constant. In fact, if $\beta_1, \cdots, \beta_m, \dfrac{2\pi}{p}$ satisfied two linear relations of the form (10) and not proportional to one another, we could, by elimination of $\dfrac{2\pi}{p}$, obtain a non-trivial linear relation with rational coefficients between $\beta_1, \cdots, \beta_m$ in contradiction to our assumption that $\beta_1, \cdots, \beta_m$ are rationally independent. In this case Kronecker's theorem states that the inequalities (9) have solutions in τ if and only of $n_1\xi_1 + \cdots + n_m\xi_m$ is an integral multiple of 2π. This implies that $g(\boldsymbol{u})$ has the form $h(n_1u_1 + \cdots + n_mu_m)$ where $h(u)$ has the period 2π. This proves the lemma.

Definition 15. *An almost periodic movement in the metric space* $\mathfrak{M}$ *is called reducible if it is homotopic to a periodic movement. If it is not reducible, it is called irreducible.*

Definiton 16. *A continuous function* $x = g(\boldsymbol{u}) = g(u_1, \cdots, u_m)$, $-\infty < u_\nu < \infty$, $\nu = 1, \cdots, m$, $x \varepsilon \mathfrak{M}$ *with the period* $2N\pi$, N *integral, in each variable is called an m-dimensional torus mapping into* $\mathfrak{M}$. *A continuous function* $x = g(\boldsymbol{u}; v) = g(u_1, \cdots, u_m; v)$, $-\infty < u_\nu < \infty$, $\nu = 1, \cdots, m$, $\alpha \leq v \leq \beta$, $x \varepsilon \mathfrak{M}$, *with the period* $2N\pi$, N *integral, in the variables* $u_1, \cdots, u_m$ *is called a family of torus mappings into* $\mathfrak{M}$. *Two torus mappings* $g_1(\boldsymbol{u})$ *and* $g_2(\boldsymbol{u})$ *into* $\mathfrak{M}$ *are called homotopic if there exists a family* $g(\boldsymbol{u}; v)$, $0 \leq v \leq 1$, *of torus mappings into* $\mathfrak{M}$ *such that* $g(\boldsymbol{u}; 0) = g_1(\boldsymbol{u})$ *and* $g(\boldsymbol{u}; 1) = g_2(\boldsymbol{u})$. *A torus mapping into* $\mathfrak{M}$ *is called reducible if it is homotopic to a torus mapping of the form* $h(\chi(\boldsymbol{u}))$ *where* $\chi = \chi(\boldsymbol{u})$ *is a real-valued, continuous function with the property that* $e^{i\chi(\boldsymbol{u})}$ *has the period* $2N\pi$ *in each variable, while* $h(\chi)$ *is a continuous function with the period* 2π. *A torus mapping which is not reducible is called irreducible.*

Briefly, a torus mapping is reducible if it can be contracted to a torus mapping into a closed curve in $\mathfrak{M}$. By the following lemma the question about reducibility of an almost periodic movement in $\mathfrak{M}$ is reduced to the question about reducibility of a torus mapping. The proof depends on results from Chapter 1.

Lemma 22. *Let* $g_1(\boldsymbol{u})$ *and* $g_2(\boldsymbol{u})$ *be two m-dimensional torus mappings into* $\mathfrak{M}$ *and let* $\beta_1, \cdots, \beta_m$ *denote rationally independent real numbers. The almost periodic movements* $g_1(\boldsymbol{\beta}t)$ *and* $g_2(\boldsymbol{\beta}t)$ *are homotopic if and only if* $g_1(\boldsymbol{u})$ *and* $g_2(\boldsymbol{u})$ *are homotopic torus mappings.*

Proof. If $g_1(\boldsymbol{u})$ and $g_2(\boldsymbol{u})$ are homotopic, there exists a family $g(\boldsymbol{u}; v)$, $0 \leq v \leq 1$ of torus mappings into $\mathfrak{M}$. The function $g(\boldsymbol{u}; v) = g(u_1, \cdots, u_m; v)$ has a period $2N\pi$ in each of the variables u_ν and it satisfies $g(\boldsymbol{u}; 0) = g_1(\boldsymbol{u})$ and $g(\boldsymbol{u}; 1) = g_2(\boldsymbol{u})$. According to Theorem 4, $g(\boldsymbol{\beta}t; v)$ is a uniformly continuous family of almost periodic functions; and, since $g(\boldsymbol{\beta} t; 0) = g_1(\boldsymbol{\beta} t)$ and $g(\boldsymbol{\beta} t; 1) = g_2(\boldsymbol{\beta} t)$, this proves that $g_1(\boldsymbol{\beta} t)$ and $g_2(\boldsymbol{\beta} t)$ are homotopic.

If, on the other hand, $g_1(\boldsymbol{\beta} t)$ and $g_2(\boldsymbol{\beta} t)$ are homotopic, there exists a uniformly continuous family $f(t; v)$, $0 \leq v \leq 1$, of almost periodic movements satisfying $f(t; 0) = g_1(\boldsymbol{\beta} t)$ and $f(t; 1) = g_2(\boldsymbol{\beta} t)$. According to Theorem 5 the family $f(t; v)$ has a basis, and we can obviously use $\beta_1, \cdots, \beta_m$ as the first numbers of this basis such that the basis is $\beta_1, \beta_2, \cdots$. According to Theorem 6 the family $f(t; v)$ has a spatial extension $G(\boldsymbol{u}; v) = G(u_1, \cdots, u_m; v)$ and $G(\boldsymbol{\beta} t; v) = f(t; v)$. From Lemma 13 it follows that $G(\boldsymbol{u}; 0) = g_1(\boldsymbol{u})$ and $G(\boldsymbol{u}; 1) = g_2(\boldsymbol{u})$. These properties of the family $G(\boldsymbol{u}; v)$ are preserved if we give u_{m+1}, $u_{m+2}, \cdots$ arbitrary constant values. We may therefore assume that $G(\boldsymbol{u}; v) = G(u_1, \cdots, u_m; v)$. Let Δ be the real number introduced in Definition 12. According to Theorem 9 there exist an integer N and a continuous function $h(\boldsymbol{u}; v) = h(u_1, \cdots, u_m; v)$, $0 \leq v \leq 1$, such that $[h(\boldsymbol{u}; v), G(\boldsymbol{u}; v)] \leq \Delta$ and the functions $h(\boldsymbol{u}; v)$, $g_1(\boldsymbol{u})$ and $g_2(\boldsymbol{u})$ have the period $2N\pi$ in each of the variables u_ν. By means of the function φ of Definition 12 we define

$$g(\boldsymbol{u}; v) = \varphi(g_1(\boldsymbol{u}); 3v; h(\boldsymbol{u}; 0)), \qquad 0 \leq v \leq \frac{1}{3}$$

$$g(\boldsymbol{u}; v) = h(\boldsymbol{u}; 3v - 1), \qquad \frac{1}{3} \leq v \leq \frac{2}{3}$$

$$g(\boldsymbol{u}; v) = \varphi(h(\boldsymbol{u}; 1); 3v - 2; g_2(\boldsymbol{u})), \qquad \frac{2}{3} \leq v \leq 1.$$

The function $g(\boldsymbol{u}; v)$ is continuous and it has the period $2N\pi$ in each of the variables u_ν. It satisfies the conditions $g(\boldsymbol{u}; 0) = g_1(\boldsymbol{u})$, $g(\boldsymbol{u}; 1) = g_2(\boldsymbol{u})$. Hence $g_1(\boldsymbol{u})$ and $g_2(\boldsymbol{u})$ are homotopic. This completes the proof.

Lemma 23. *If a torus mapping $g_1(\boldsymbol{u}) = g_1(u_1, \cdots, u_m)$ is homotopic to a torus mapping $g_2(\boldsymbol{u}) = g_2(u_1, \cdots, u_{m-1})$, it is also homotopic to the torus mapping $g_1(u_1, \cdots, u_{m-1}, k)$ where k is an arbitrary constant.*

Proof. There exists a family $g(\boldsymbol{u}; v)$, $0 \leq v \leq 1$, of torus mappings satisfying $g(\boldsymbol{u}; 0) = g_1(\boldsymbol{u})$ and $g(\boldsymbol{u}; 1) = g_2(\boldsymbol{u})$.

It follows that $g(u_1, \cdots, u_{m-1}, k; v)$ is a family of torus mappings, which proves that $g_1(u_1, \cdots, u_{m-1}, k)$ is homotopic to

$$g(u_1, \cdots, u_{m-1}, k; 1) = g_2(u_1, \cdots, u_m).$$

Theorem 11. *Let $f(t)$ be an almost periodic movement in a complete and continuously locally arcwise connected space $\mathfrak{M}$ and let $\beta_1, \beta_2, \cdots$ be a basis of $f(t)$. There exists a finite subset $\beta_{n_1}, \cdots, \beta_{n_q}$ of the basis such that the following conditions are satisfied.*

1. *$f(t)$ is homotopic to a function $g(\beta_{n_1}t, \cdots, \beta_{n_q}t)$ where $g(u_1, \cdots, u_q)$ is continuous and has the period $2N\pi$ (N* integral) *in each variable.*

2. *If a subset B of the basis $\beta_1, \beta_2, \cdots$ is a basis for an almost periodic movement homotopic to $f(t)$, then B contains the numbers $\beta_{n_1}, \cdots, \beta_{n_q}$.*

Proof. According to Lemma 20 the movement $f(t)$ is homotopic to a diagonal movement $g_1(\beta_1 t, \cdots, \beta_m t)$ where $g_1(\boldsymbol{u})$ has a period $2N_1\pi$ in each variable. We cannot be certain that $g_1(\boldsymbol{u})$ has the period 2π, because the basis numbers $\beta_1, \cdots, \beta_m$ are given beforehand, but we know from Theorems 6 and 9 that $f(t)$ can be approximated to any given accuracy by a function $g^*\left(\frac{\beta_1}{N_1}t, \cdots, \frac{\beta_m}{N_1}t\right)$ where $g^*(\boldsymbol{u})$ has the period 2π in each variable. This justifies our use of Lemma 20. We shall call a variable u_ν inessential for $g_1(\boldsymbol{u})$ if $g_1(\boldsymbol{u})$ is homotopic to a torus mapping which is independent of u_ν. A variable u_μ is called essential for $g_1(\boldsymbol{u})$ if it is not inessential. Let $u_{n_1}, \cdots, u_{n_q}$ denote the essential variables. From Lemma 23 follows that $g_1(\boldsymbol{u})$ is homotopic to a function $g(u_{n_1}, \cdots, u_{n_q})$ depending only on the essential variables and with a period $2N\pi$ in each variable. By Lemma 22 this implies that $g_1(\boldsymbol{\beta}t)$ is homotopic to $g(\boldsymbol{\beta}t)$, hence, that $f(t)$ is homotopic to $g(\boldsymbol{\beta}t) = g(\beta_{n_1}t, \cdots, \beta_{n_q}t)$. If, on the other hand, $f(t)$ is homotopic to a movement with basis B, it is, according to Lemma 20, homotopic to a movement $g_2(\beta_{\nu_1}t, \cdots, \beta_{\nu_p}t)$ where $\beta_{\nu_1}, \cdots, \beta_{\nu_p}$ belong to B and $g_2(\boldsymbol{u}) = g_2(u_{\nu_1}, \cdots, u_{\nu_p})$ has a period $2N_2\pi$ in each variable. By Lemma 22 it follows that $g(u_{n_1}, \cdots, u_{n_q})$ is homotopic to

$g_2(u_{\nu_1}, \cdots, u_{\nu_p})$ and by Lemma 23 that $g(\boldsymbol{u})$ is homotopic to a function depending only on the variables u_{n_μ} occurring among the variables u_{ν_μ}. But the variables u_{n_μ} were essential. Hence every u_{n_μ} occurs among $u_{\nu_1}, \cdots, u_{\nu_p}$, i. e. every β_{n_μ} belongs to B. This completes the proof.

Lemma 24. *Let $g_1(\boldsymbol{u})$ be an m-dimensional torus mapping into $\mathfrak{M}$ and let $\beta_1, \cdots, \beta_m$ be rationally independent numbers. If the diagonal movement $f_1(t) = g_1(\boldsymbol{\beta} t)$ is homotopic to a movement $f_2(t)$ with the period p, and the numbers $\beta_1, \cdots, \beta_m, \frac{2\pi}{p}$ are rationally independent, then $f_1(t)$ is homotopic to a constant.*

Proof. We choose the extra basis number $\beta_{m+1} = \frac{2\pi}{p}$. According to Lemma 13 $f_1(t)$ and $f_2(t)$ have $g_1(\boldsymbol{u})$ and $f_2\left(\frac{p}{2\pi} u_{m+1}\right)$ as spatial extensions. By Lemma 23 this implies that $g_1(\boldsymbol{u})$ is homotopic to a constant and the lemma follows from Lemma 22.

Theorem 12. *Let $g(\boldsymbol{u})$ be an m-dimensional torus mapping into a complete and continuously locally arcwise connected space $\mathfrak{M}$ and let $\beta_1, \cdots, \beta_m$ be rationally independent numbers. The almost periodic movement $f(t) = g(\boldsymbol{\beta} t)$ is reducible if and only if the torus mapping $g(\boldsymbol{u})$ is reducible.*

Proof. If $f(t)$ is reducible, it is homotopic to a movement $f_1(t)$ with a period p. If $\beta_1, \cdots, \beta_m, \frac{2\pi}{p}$ are rationally independent, it follows from Lemma 24 that $f(t)$ is homotopic to a constant. Hence, by Lemma 22, $g(\boldsymbol{u})$ is homotopic to a constant, i. e. reducible. If $\beta_1, \cdots, \beta_m, \frac{2\pi}{p}$ are rationally dependent, we have a relation

$$n \frac{2\pi}{p} = n_1 \beta_1 + \cdots + n_m \beta_m$$

where $n, n_1, \cdots, n_m$ are integers and $n \neq 0$. The numbers $\frac{\beta_1}{n}, \cdots, \frac{\beta_m}{n}$ form a finite integral basis (Definitions 10 and 11) of $f_1(t)$. According to the Theorems 7 and 8, the movement $f_1(t)$ has a spatial extension $g_1(\boldsymbol{u}) = g_1(u_1, \cdots, u_m)$ with the period 2π in each variable, and we have

$$f_1(t) = g_1\left(\frac{\beta_1}{n}t, \cdots, \frac{\beta_m}{n}t\right).$$

According to Lemma 22 the function $g(\boldsymbol{u})$ is homotopic to $g_1\left(\frac{u_1}{n}, \cdots, \frac{u_m}{n}\right)$. Since $g_1(\beta_1 t, \cdots, \beta_m t)$ has the period $\frac{n}{p}$, Lemma 21 implies that $g_1(\boldsymbol{u}) = h(n_1 u_1 + \cdots + n_m u_m)$ where $n_1, \cdots, n_m$ are integers and $h(u)$ has the period 2π. This proves that the torus mapping $g(\boldsymbol{u})$ is reducible.

If, on the other hand, $g(\boldsymbol{u})$ is reducible, it is homotopic to a torus mapping $h(\chi(\boldsymbol{u}))$ where $\chi(\boldsymbol{u})$ is a real-valued, continuous function with the property that $e^{i\chi(\boldsymbol{u})}$ has the period $2N\pi$, N integral, in each variable, while $h(\chi)$ is a continuous function with the period 2π.

Since $\chi(\boldsymbol{u})$ is the argument of the periodic function $e^{i\chi(\boldsymbol{u})}$, it has the form ([5])

$$\chi(\boldsymbol{u}) = \frac{n_1}{N}u_1 + \cdots + \frac{n_m}{N}u_m + \psi(\boldsymbol{u})$$

where $n_1, \cdots, n_m$ are integers, and $\psi(\boldsymbol{u})$ is a continuous function with the period $2N\pi$ in each variable. According to Lemma 22, $f(t)$ is homotopic to the almost periodic movement

$$h(\chi(\boldsymbol{\beta}t)) = h(\gamma t + \psi(\boldsymbol{\beta}t)) \tag{11}$$

where

$$\gamma = \frac{n_1\beta_1 + \cdots + n_m\beta_m}{N}.$$

The family

$$h(\gamma t + (1 - v)\psi(\boldsymbol{\beta}t)),\ 0 \leqq v \leqq 1, \tag{12}$$

is obviously a uniformly continuous family of almost periodic movements. In fact, it is equal to $H(\boldsymbol{\beta}t; v)$ where

$$H(\boldsymbol{u}; v) = h\left(\frac{n_1}{N}u_1 + \cdots + \frac{n_m}{N}u_m + (1 - v)\psi(\boldsymbol{u})\right)$$

is a continuous function with the period $2N\pi$ in each variable. The family (12) proves that the movement (11) is homotopic to the function $h(\gamma t)$ which has the period $\frac{2\pi}{\gamma}$. This completes the proof of Theorem 12.

Theorem 13. *Every almost periodic movement in a complete and continuously locally arcwise connected space $\mathfrak{M}$ is reducible if and only if every torus mapping into $\mathfrak{M}$ is reducible.*

Proof. The theorem follows immediately from Lemma 20 and Theorem 12.

3. Almost Periodic Movements on a Sphere.

In this section we consider the m-dimensional sphere, i. e. the set of real vectors $\boldsymbol{x} = (x_0, \cdots, x_m)$ satisfying the condition

$$|\boldsymbol{x}|^2 = x_0^2 + \cdots + x_m^2 = 1 .$$

We shall use the ordinary vector notations. If $\boldsymbol{y} = (y_0, \cdots, y_m)$ is another vector, we define $\boldsymbol{x} \pm \boldsymbol{y} = (x_0 \pm y_0, \cdots, x_m \pm y_m)$ and $\boldsymbol{xy} = x_0 y_0 + \cdots + x_m y_m$. If λ is a real number, we define further $\lambda \boldsymbol{x} = (\lambda x_0, \cdots, \lambda x_m)$. We consider the m-dimensional sphere as a metric space $\mathfrak{S}_m$ with the metric

$$[\boldsymbol{x}, \boldsymbol{y}] = |\boldsymbol{y} - \boldsymbol{x}| = \sqrt{(y_0 - x_0)^2 + \cdots + (y_m - x_m)^2}.$$

Obviously we have

Lemma 25. *The metric space $\mathfrak{S}_m$ is complete and continuously locally arcwise connected.*

Lemma 26. *There exists an irreducible torus mapping into $\mathfrak{S}_m$ if $m \geq 2$.*

Proof. Let $\mathfrak{D}_m$ denote the parallellepiped $|u_\nu| \leq \pi$, $\nu = 1, \cdots, m$. If we identify all boundary points of this parallellepiped with one another ([1] p. 64), it becomes an m-dimensional sphere and the identical mapping becomes a mapping of $\mathfrak{D}_m$ onto this sphere. This mapping can be extended to a continuous function with the period 2π in every u_ν, i. e. to a torus mapping. Since the mapping is one-to-one in the interior of D_m, its degree is ± 1. Consequently this mapping is irreducible.

Theorem 14. *There exists an irreducible almost periodic movement in $\mathfrak{S}_m$ if $m \geqq 2$.*

Proof. Immediate consequence of Theorem 13 and Lemma 26.

In the case $m = 2$ we shall give some examples of irreducible almost periodic movements by explicit expressions or by geometrical descriptions. In the proof of Lemma 26 we used a torus mapping which mapped the boundary of the parallellepiped $|u_\nu| \leq \pi$, $\nu = 1, \cdots, m$, onto a single point of the sphere. In

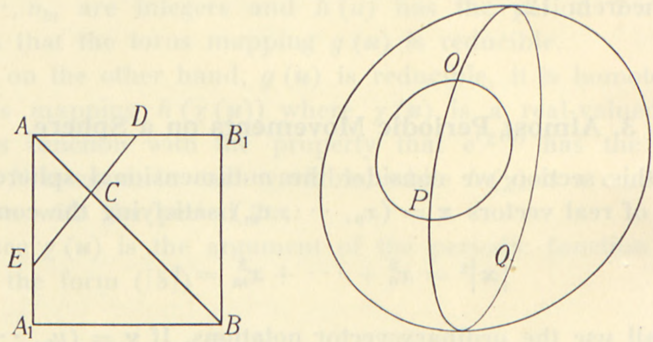

Fig. 1.

the case $m = 2$ we shall map the square $|u_1| \leq \pi$, $|u_2| \leq \pi$ on the sphere $x_1^2 + x_2^2 + x_3^2 = 1$ such that the boundary of the square is mapped onto one point of the sphere.

To accomplish this we map the diagonal AB (fig. 1) onto a great circle OPQ on the sphere such that A and B are mapped into O, and such that corresponding lengths are proportional. If γ is an irrational number, the square is covered by segments with the slope γ, as for instance DCE. We map the segment DCE into the sphere onto a circle perpendicular to OPQ. The images of D, C, and E are already chosen and each of the segments CD and CE is mapped onto a semicircle such that corresponding lengths are proportional. This yields a mapping with the desired properties. If we put $u_1 = t$, $u_2 = a + \gamma t$ where a is constant, we obtain a very simple irreducible almost periodic movement, which can be described in the following way: On a great circle (fig. 2) we choose a point O and an infinite sequence $\cdots, P_{-1}, P_0, P_1, \cdots$ of points such that the arcs $P_{\nu-1}P_\nu$ have the same length, and the ratio between this length and the circumference is irrational. The movement describes the circles with diameters $\cdots, OP_{-1}, OP_0, OP_1, \cdots$ in succession according to the indices.

The speed is constant on each semicircle, but it varies a little from one semicircle to another. However, it is not difficult to prove that the movement which runs through the circles with constant speed all the time is also almost periodic and irreducible.

The mapping of the square $|u_1| \leq \pi$, $|u_2| \leq \pi$ onto the sphere can also be carried out such that the side A_1B corresponds to

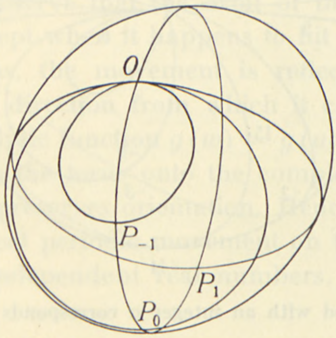

Fig. 2.

a great circle and the perpendiculars to A_1B correspond to small circles through a point O of the great circle and perpendicular to it. In this way we obtain the following movement

$$(13) \quad \begin{cases} x_1 = \left|\sin\frac{\beta_1}{2}t\right| \sin\beta_2 t \\ x_2 = \sin\frac{\beta_1}{2}t \cos\frac{\beta_1}{2}t\,(1-\cos\beta_2 t) \\ x_3 = 1-\sin^2\frac{\beta_1}{2}t\,(1-\cos\beta_2 t) \end{cases}$$

where β_1 and β_2 are rationally independent. We have written $\frac{\beta_1}{2}$ instead of β_1, because (β_1, β_2) then is an integral basis. We shall omit the (not very interesting) proof of the formulas (13). Fig. 3 shows part of the projection on the (x_1, x_2)-plane of the orbit of the movement.

We can construct an irreducible mapping of the torus onto the sphere in a quite different way. We divide the square $|u_1| \leq \pi$,

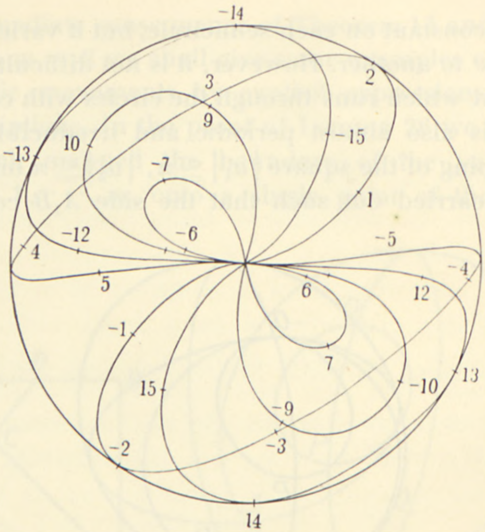

Fig. 3. The point marked with an integer n corresponds to $t = \frac{n\pi}{4}$. The values $n = -16, -8, 0, 8,$ and 16 are in the center. The values $n = \pm 11$ are marked near the center.

$|u_2| \leq \pi$ into $4\,pq$ congruent rectangles by $2\,p-1$ lines parallel to one pair of its sides and $2\,q-1$ lines parallel to the other pair. We fold the square along all these lines. Points that are brought together by the folding of the square are identified if

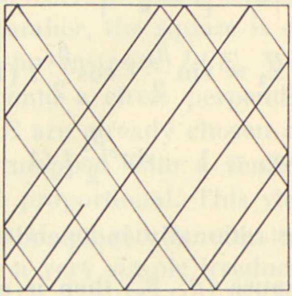

Fig. 4.

they are given the same colour when the $4\,pq$ rectangles are coloured alternately black and white like the squares on a chess-board. In this way we obtain a very simple irreducible mapping of the torus onto the sphere. In the simplest case, $p = q = 1$, we have drawn the corresponding irreducible almost periodic movement in fig. 4. The "sphere" consists of two square sheets, joined along the edges. The movement follows the orbit of a billiard ball, started inside the square in a direction with irrational slope against the sides, and reflected from the sides according to the law of equal angles. By each reflexion the almost periodic movement passes into the other sheet. The movement has constant speed. We observe that the orbit of the movement never intersects itself except when it happens to hit one of the corners. When this happens, the movement is reflected into the same sheet and into the direction from which it came.

An arbitrary elliptic function $g(w) = g(u_1 + iu_2)$ with periods 2π and $2\pi i$ maps the torus onto the complex number sphere, and the mapping preserves orientation. Hence $g((\beta_1 + i\beta_2)t)$ is an irreducible almost periodic movement on the sphere if β_1 and β_2 are rationally independent real numbers.

4. Almost Periodic Movements in a Projective Space.

Let $\mathfrak{S}_m$ denote the m-dimensional sphere. The projective space $\mathfrak{P}_m$ with the elliptic non-Euclidean metric consists of all points $(\boldsymbol{x}, -\boldsymbol{x}) = (-\boldsymbol{x}, \boldsymbol{x})$ where $\boldsymbol{x}$ belongs to the sphere, the distance being defined by

$$[(\boldsymbol{x}, -\boldsymbol{x}), (\boldsymbol{y}, -\boldsymbol{y})] = \mathrm{Min}\,([\boldsymbol{x}, \boldsymbol{y}], [\boldsymbol{x}, -\boldsymbol{y}]).$$

It is well known that $\mathfrak{P}_m$ is a complete metric space. If

$$(\boldsymbol{x}, -\boldsymbol{x}) = (f(t), -f(t)) \tag{14}$$

is a continuous movement in $\mathfrak{P}_m$, we can obviously choose the representation (14) such that $f(t)$ is a continuous movement in $\mathfrak{S}_m$. If

$$(\boldsymbol{x}, -\boldsymbol{x}) = (f(t; v), -f(t; v))$$

is a uniformly continuous family of almost periodic movements in $\mathfrak{P}_m$, we can assume that $f(t; v)$ is continuous. If E denotes the mapping $E(\boldsymbol{x}) = -\boldsymbol{x}$ of $\mathfrak{S}_m$ onto itself and τ is an ε-translation number of $(f(t; v), -f(t; v))$, we have

$$[f(t; v), E^{\nu} f(t + \tau; v)] \leq \varepsilon$$

for some power ν. It follows that

$$[f(t; v), E^{2\nu} f(t + 2\tau; v)] \leq 2\varepsilon.$$

But $E^{2\nu}$ is the identity. Hence $f(t; v)$ is a uniformly continuous family of almost periodic movements in $\mathfrak{S}_m$. If $f(t)$ is an irreducible almost periodic movement in $\mathfrak{S}_m$, it follows immediately that $(f(t), -f(t))$ is an irreducible almost periodic movement in $\mathfrak{P}_m$. We have thus proved the theorem

Theorem 15. *There exists an irreducible almost periodic movement in the projective space* $\mathfrak{P}_m$ *if* $m \geq 2$.

5. Some Remarks Concerning the General Problem.

The question of homotopy between m-dimensional tori in a general metric space $\mathfrak{M}$ has never been investigated. If the investigations are restricted to tori which are topological products of a fixed $m-1$-dimensional torus in $\mathfrak{M}$ and a closed curve, it is possible, as R. H. Fox [9] pointed out, to introduce general torus homotopy groups analogous to the ordinary homotopy groups. The particular case, where the tori are "pinched", i. e. where the given $m-1$-dimensional torus reduces to a single point, was investigated in detail by R. H. Fox, who proved that the "pinched" tori homotopy groups can be built from the ordinary homotopy groups by means of Whitehead products. It follows in particular from his investigations that every "pinched" torus in $\mathfrak{M}$ is reducible if and only if every sphere in $\mathfrak{M}$ is trivial, i. e., homotopic to a point.

Let a and b denote two elements of the fundamental group of $\mathfrak{M}$ corresponding to a point P. They correspond to two curves A and B in $\mathfrak{M}$ starting and ending at P. If a and b are commutative, the curve composed of A, B, $-A$, and $-B$ can be contracted to a point and by this contraction it describes a torus. If every torus in $\mathfrak{M}$ is reducible, we can conclude that some multiple of a is homotopic to some multiple of b. It follows that *every commutative subgroup of the fundamental group is cyclic.* This is a necessary but not a sufficient condition for reducibility of every torus in $\mathfrak{M}$. If, however, the further condition that every m-dimensional sphere in $\mathfrak{M}$ is trivial for $m > 1$, is satisfied, it is not hard to prove that every torus can be contracted to a "pinched" torus, and it follows then from the results of R. H. Fox that every torus in $\mathfrak{M}$ is reducible. Hence, we have the general theorem:

Every almost periodic movement in a complete and continuously arcwise locally connected space $\mathfrak{M}$ is reducible if the mth *homotopy group of $\mathfrak{M}$ is trivial for $m > 1$, and every Abelian subgroup of the fundamental group of $\mathfrak{M}$ is cyclic.*

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